

ZEBRA-PERCOLATION ON CAYLEY TREES

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ABSTRACT. We consider Bernoulli (bond) percolation with parameter p on the Cayley tree of order k . We introduce the notion of zebra-percolation that is percolation by paths of alternating open and closed edges. In contrast with standard percolation with critical threshold at $p_c = 1/k$, we show that zebra-percolation occurs between two critical values $p_{c,1}$ and $p_{c,2}$ (explicitly given). We provide the specific formula of zebra-percolation function.

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1. INTRODUCTION AND DEFINITIONS

Percolation on trees still remains the subject of many open problems. The purpose of this paper is to study the percolation phenomenon by paths of alternating open and closed bonds. Such paths are called zebra-paths.

We consider the Cayley tree $\Gamma^k = (V, L)$ where each vertex has $k + 1$ neighbors with V being the set of vertices and L the set of bonds. Bonds are independently open with probability p (and closed with probability $1 - p$). We let P_p be corresponding probability measure.

On this tree we fix a given vertex e (the root) and consider the following event

$$E = \{\text{An infinite zebra-path contains the root}\}. \quad (1.1)$$

By path we mean a collection of consecutive bonds (appearing only once) sharing a common endpoint. The zebra-percolation function is defined by

$$\zeta_k(p) = P_p(E). \quad (1.2)$$

The paper is organized as follows. In Section 2 we show that zebra-percolation occurs in the range $p \in (p_{c,1}, p_{c,2})$. This holds as soon as $k \geq 3$ and the two critical values are explicitly given. Section 3 is devoted to standard percolation. In Section 4 we give a relation between standard percolation and zebra-percolation. The last section is devoted to some discussions and open problems.

2. TWO CRITICAL VALUES

The existence of two critical values is a consequence of the following dichotomy

Theorem 1. *The zebra percolation function satisfies*

- 1) *If $k^2 p(1 - p) < 1$, then $\zeta_k(p) = 0$.*

2) If $k^2p(1-p) > 1$, then $\zeta_k(p) > 0$.

Proof. 1) Consider on the tree Γ^k all paths of length n starting from the root. We will denote hereafter by W_n the set of endpoints of these paths (excluding the root). Let \mathcal{F}_n be the event that there is a zebra-path of length n . The probability \mathcal{P}_n for such an event is

$$\mathcal{P}_n = \begin{cases} 2(p(1-p))^{n/2}, & \text{if } n \text{ is even} \\ (p(1-p))^{(n-1)/2}, & \text{if } n \text{ is odd.} \end{cases} \quad (2.1)$$

The number of paths is at most $|W_n| = (k+1)k^{n-1}$. This implies that

$$P_p(\mathcal{F}_n) \leq 2(k+1)k^{n-1}(p(1-p))^{[n/2]},$$

which, under the condition $k^2p(1-p) < 1$, goes to 0 as $n \rightarrow \infty$. Hereafter $[\cdot]$ denotes the integer part. We then get $\zeta_k(p) = 0$.

2) We shall show that if $k^2p(1-p) > 1$, then the root zebra-percolates with positive probability. Let X_n denote the number of vertices belonging to W_n and zebra-connected to the root. We will apply the method of second moment to the random variable X_n (see, e.g. [6]). We have

$$P(X_n > 0) \geq \frac{E[X_n]^2}{E[X_n^2]}. \quad (2.2)$$

By linearity, we have that $E(X_n) = |W_n|\mathcal{P}_n$. If we can show that for some constant M and for all n ,

$$E(X_n^2) \leq ME(X_n)^2, \quad (2.3)$$

we would then have that $P_p(X_n > 0) \geq \frac{1}{M}$ for all n . The events $\{X_n > 0\}$ are decreasing and so countable additivity yields $P_p(X_n > 0, \forall n) \geq \frac{1}{M}$. But the latter event is the same as the event that the root is percolating and one is done. We now bound the second moment in order to establish (2.3). Letting $U_{v,w}$ be the event that both v and w are zebra-connected to the root, we have that

$$E(X_n^2) = \sum_{v,w \in W_n} P_p(U_{v,w}). \quad (2.4)$$

Now $P_p(U_{v,w}) = \mathcal{P}_n^2 \mathcal{P}_{m_{v,w}}^{-1}$, where $m_{v,w}$ is the level at which paths from e to v and to w split. For a given v and m , the number of w with $m_{v,w}$ being m is at most $|W_n|/|W_m|$. Hence

$$E(X_n^2) \leq |W_n| \sum_{m=0}^n \mathcal{P}_n^2 \mathcal{P}_m^{-1} |W_n|/|W_m| = E(X_n)^2 \sum_{m=0}^n \frac{1}{\mathcal{P}_m |W_m|}. \quad (2.5)$$

If $\sum_{m=0}^{\infty} \frac{1}{\mathcal{P}_m |W_m|} < \infty$, then we would have (2.3). If $k^2p(1-p) > 1$, then using formula (2.1) one can see that $\frac{1}{\mathcal{P}_m |W_m|}$ decays exponentially like $(k^2p(1-p))^{-m/2}$ giving the desired convergence. \square

This theorem gives two critical values for the zebra-percolation which are solutions to $k^2p(1-p) = 1$:

$$p_{c,1}(k) = \frac{k - \sqrt{k^2 - 4}}{2k}, \quad p_{c,2}(k) = \frac{k + \sqrt{k^2 - 4}}{2k}.$$

Note that if $k \geq 3$, $0 < p_{c,1}(k) < \frac{1}{k} < \frac{1}{2} < p_{c,2}(k) < 1$. Moreover $p_{c,1}(k) + p_{c,2}(k) = 1$. This tells that $p_{c,1}$ and $p_{c,2}$ are symmetric with respect to $1/2$. When $k = 2$, $p_{c,1}(k) = p_{c,2}(k) = 1/2$ so that no zebra-percolation occurs.

3. ON PERCOLATION FUNCTION

Consider standard percolation model on a Cayley tree. Denote by $\theta_k(p)$ the standard percolation function, that is the probability with respect to P_p that there exists an infinite cluster of open edges containing the root. We refer the reader to [2], [3], [4], [5].

Proposition 1. *The function $\theta_k(p)$ satisfies*

$$\theta_k(p) = \begin{cases} 0, & \text{if } p \leq \frac{1}{k} \\ \hat{\theta}_k(p), & \text{if } p > \frac{1}{k}, \end{cases}$$

where $\hat{\theta}_k(p)$ is a unique solution to the following functional equation

$$\hat{\theta}_k(p) = 1 - \left(1 - p\hat{\theta}_k(p)\right)^k, \quad p > \frac{1}{k}. \quad (3.1)$$

Proof. Let e be the root of the Cayley tree, and $S(e)$ the set of direct successors of the root. Denote by \mathcal{A}_i the event that vertex $i \in S(e)$ is in an infinite component, which is not connected to e . Then by self-similarity we get

$$P_p(\mathcal{A}_i) = \theta_k(p), \quad \text{for any } i \in S(e).$$

Let \mathcal{B}_i be the event that the edge $\langle e, i \rangle$ is open and \mathcal{A}_i holds. Then

$$P_p(\mathcal{B}_i) = p\theta_k(p), \quad \text{for any } i \in S(e).$$

Since $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ are independent, using inclusion-exclusion principle, we get

$$\begin{aligned} \theta_k(p) &= P_p\left(\bigcup_{i=1}^k \mathcal{B}_i\right) = \\ &= \sum_{i=1}^k P_p(\mathcal{B}_i) - \sum_{\substack{i,j: \\ i < j}} P_p(\mathcal{B}_i \cap \mathcal{B}_j) + \sum_{\substack{i,j,q: \\ i < j < q}} P_p(\mathcal{B}_i \cap \mathcal{B}_j \cap \mathcal{B}_q) - \dots + (-1)^{k-1} P_p\left(\bigcap_{i=1}^k \mathcal{B}_i\right) = \\ &= kp\theta_k(p) - \binom{k}{2}(p\theta_k(p))^2 + \binom{k}{3}(p\theta_k(p))^3 - \dots + (-1)^{k-1}(p\theta_k(p))^k = \\ &= 1 - (1 - p\theta_k(p))^k. \end{aligned}$$

Hence $\theta_k(p)$ is a fixed point of the function

$$f(x) = 1 - (1 - px)^k, \quad x \in [0, 1].$$

The proof is then completed by using the following lemma: □

Lemma 1. *The function f satisfies*

- i. *If $p \leq \frac{1}{k}$ then the function $f(x)$ has a unique fixed point 0.*
- ii. *If $p > \frac{1}{k}$ then the function $f(x)$ has two fixed points 0 and $\hat{\theta}$.*

Proof. Note that 0 is a fixed point of f . On the other hand, $f(1) = 1 - (1 - p)^k$ and

$$f'(x) = kp(1 - px)^{k-1} \geq 0, \quad f''(x) = -k(k-1)p^2(1 - px)^{k-2} \leq 0, \quad x \in [0, 1].$$

Hence f is increasing and concave. It is easy to see that f has a unique fixed point $\hat{\theta} \in (0, 1]$ when $f'(0) = kp > 1$ and no fixed point when $f'(0) = kp \leq 1$. This completes the proof. \square

Simple computations show that

$$\theta_2(p) = \begin{cases} 0, & \text{if } p \leq \frac{1}{2} \\ \frac{2p-1}{p^2}, & \text{if } p > \frac{1}{2}. \end{cases}$$

and

$$\theta_3(p) = \begin{cases} 0, & \text{if } p \leq \frac{1}{3} \\ \frac{2(3p-1)}{p(3p+\sqrt{p(4-3p)})}, & \text{if } p > \frac{1}{3}. \end{cases}$$

The general solution is given through the inverse function

Proposition 2. *The function $\hat{\theta}_k(p)$, $p > 1/k$, $k \geq 2$ is invertible with inverse*

$$\hat{\theta}_k^{-1}(p) = \frac{1 - \sqrt[k]{1-p}}{p}. \quad (3.2)$$

Proof. First we shall prove that $\hat{\theta}_k(p)$ is one-to-one. For $p_1, p_2 \in (1/k, 1)$, we get from equation (3.1)

$$\hat{\theta}_k(p_1) - \hat{\theta}_k(p_2) = \left[(p_1 - p_2)\hat{\theta}_k(p_1) + p_2 \left(\hat{\theta}_k(p_1) - \hat{\theta}_k(p_2) \right) \right] \cdot \mathcal{U}, \quad (3.3)$$

where $\mathcal{U} = \sum_{i=0}^{k-1} (1 - p_1\hat{\theta}_k(p_1))^{k-1-i} (1 - p_2\hat{\theta}_k(p_2))^i > 0$.

Since $\hat{\theta}_k(p) > 0$ for any $p > 1/k$, if $\hat{\theta}_k(p_1) = \hat{\theta}_k(p_2)$ then from equality (3.3) we get $p_1 = p_2$. Hence $\hat{\theta}_k(p)$ is one-to-one, i.e. invertible.

Solving the equation $x = 1 - (1 - px)^k$ with respect to p for $x \in [0, 1]$, we get $p = g(x) = x^{-1}(1 - \sqrt[k]{1-x})$. Now by (3.1) we have $p = g(\hat{\theta}_k(p))$ for any $p > \frac{1}{k}$. Hence g is the inverse function of $\hat{\theta}_k(p)$. \square

Note that the function $\theta_k(p)$ has following properties:

- (1) $\theta_k(p)$ is nondecreasing in p
- (2) $\theta_k(1/k) = 0$, $\theta_k(1) = 1$, $\theta_k(p) \neq 1$ for any $p < 1$
- (3) $\theta_k(p)$ is differentiable for any $p \neq 1/k$.

4. RELATION BETWEEN STANDARD AND ZEBRA PERCOLATION

Starting from the Cayley tree $\Gamma^k = (V, L)$, we construct a new tree $\hat{\Gamma}^k = (\hat{V}, \hat{L})$ as follows (see Fig. 1)

$$\hat{V} = \bigcup_{m=0}^{\infty} W_{2m}, \quad \hat{L} = \bigcup_{m=0}^{\infty} \{(x, z) : x \in W_{2m}, z \in S(y), y \in S(x)\},$$

where $S(x)$ denotes the set of direct successors of x .

It is easy to see that $\hat{\Gamma}^k$ is a regular tree of order k^2 (except on the root).

We denote by l an edge in L and by λ and edge in \hat{L} . Note that any edge $\lambda \in \hat{L}$ can be represented by two edges $l_1, l_2 \in L$, which have a common endpoint. We write this as $\lambda = (l_1, l_2)$, moreover l_1 is the closer to the root of the Cayley tree.

Now for a given configuration $\sigma \in \Omega = \{0, 1\}^L$ we define a configuration $\phi \in \Phi = \{-1, 0, +1\}^{\hat{L}}$ as the following (see Fig. 1)

$$\phi(\lambda) = \phi_{\sigma}(\lambda) = \begin{cases} -1, & \text{if } \sigma(l_1) = 0, \sigma(l_2) = 1 \\ 0, & \text{if } \sigma(l_1) = \sigma(l_2) \\ 1, & \text{if } \sigma(l_1) = 1, \sigma(l_2) = 0. \end{cases}$$

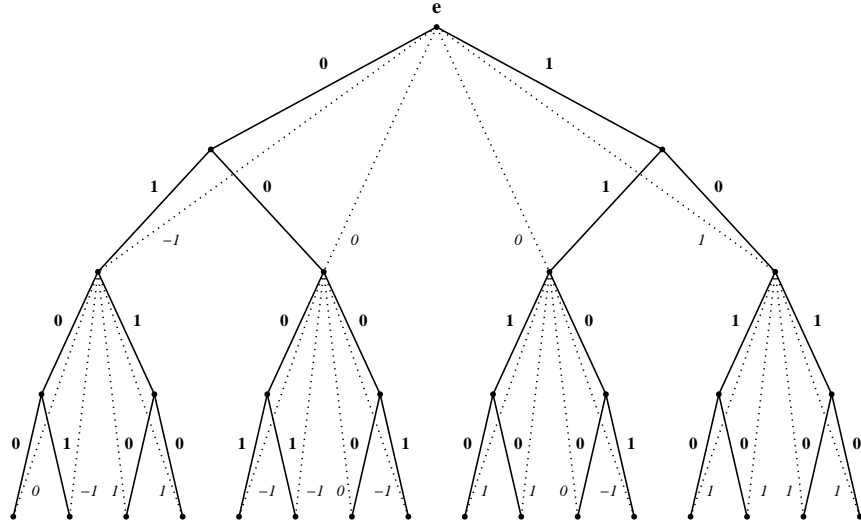


Fig. 1. Correspondence between configurations σ on Γ^2 (solid lines) and ϕ on $\hat{\Gamma}^2$ (dotted lines). A given configuration ϕ divides the set \hat{L} into clusters of (+) and (-) bonds.

We speak of the edge $\lambda \in \hat{L}$ as being open with probability q (in ϕ) if $\phi(\lambda) \neq 0$ and as being closed if $\phi(\lambda) = 0$. Let μ_q be corresponding product measure. Denote

$$\theta_{k^2}(q) = \mu_q(|\hat{C}| = \infty). \quad (4.1)$$

By our construction the following is obvious

Proposition 3. *The functions $\zeta_k(p)$ and $\theta_k(p)$ are related by*

$$\zeta_k(p) = \theta_{k^2}(p(1-p)). \quad (4.2)$$

This proposition provides an alternative proof of Theorem 1. By properties of $\theta_{k^2}(p)$ we get $\zeta_k(p) = 0$ iff $p(1-p) \leq 1/k^2$ and $\zeta_k(p) > 0$ iff $p(1-p) > 1/k^2$.

The two critical values $p_{c,1}$ and $p_{c,2}$ are the solutions of $p(1-p) = 1/k^2$.

By Proposition 3 we get

Theorem 2. *The function $\zeta_k(p)$ has the following properties:*

- a. $\zeta_k(p)$ is increasing in $p \in [0, 1/2]$, and decreasing in $p \in [1/2, 1]$.
- b. $\zeta_k(p_{c,1}) = \zeta_k(p_{c,2}) = 0$, $\max_p \zeta_k(p) = \zeta_k(1/2) = \theta_{k^2}(1/4)$.
- b. $\zeta_k(p)$ is differentiable on $[0, 1] \setminus \{p_{c,1}, p_{c,2}\}$.
- c. there is no zebra-percolation for $k = 2$.

The graphs of functions $\theta_k(p)$, $\theta_{k^2}(p)$ and $\zeta_k(p)$ are presented for $k = 3$ in Fig.2.

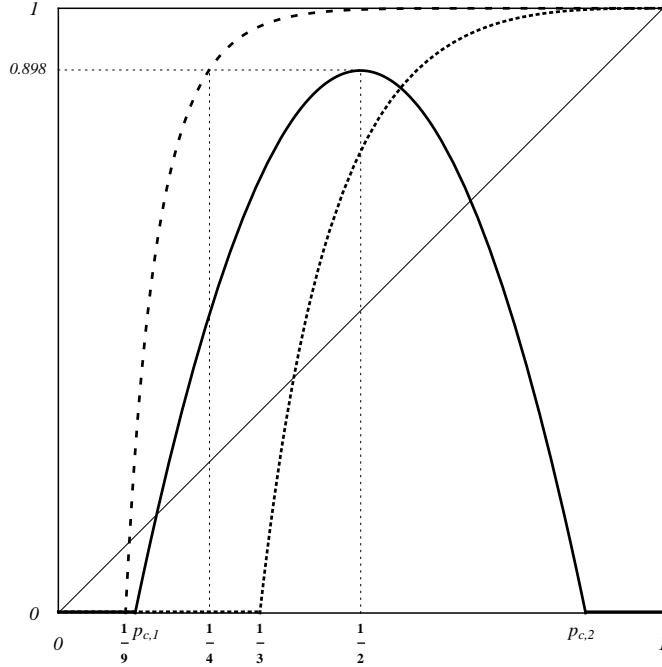


Fig. 2. Graphs of $\theta_3(p)$ (dashed line), $\theta_9(p)$ (dotted line), and $\zeta_3(p)$ (solid line).

5. OPEN PROBLEMS

An interesting problem in percolation theory is to study the distribution of the number of vertices in clusters and geometric properties of open clusters when p is close to the critical value p_c . It is believed that some of these properties are universal, i.e., depend only on the dimension of the graph. Some open problems are in order.

Problem 1. *Study distribution of the number of vertices and geometric properties of the zebra-connected clusters (made of zebra paths) when p is close to $p_{c,1}$ or $p_{c,2}$.*

It is known that \mathbb{Z}^d for large d behaves in many respects like a regular tree.

Problem 2. *Define a notion of zebra-connected component on \mathbb{Z}^d . Find the critical value(s) for zebra-percolation on \mathbb{Z}^d .*

When an infinite cluster exists, it is natural to ask how many there are (see e.g. [1]).

Problem 3. *How many infinite cluster exist for zebra-percolation ?*

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